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## BUBBLE COLLISIONS AND DEFECT FORMATION IN A DAMPING ENVIRONMENT

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### Abstract

Within the context of a first-order phase transition in the early Universe, we study the collision process for vacuum bubbles expanding in a plasma. The effects of the plasma are simulated by introducing a damping term in the equations of motion for a  $U(1)$  global field. We find that Lorentz-contracted spherically symmetric domain walls adequately describe the overdamped motion of the bubbles in the thin wall approximation, and study the process of collision and phase equilibration both numerically and analytically. With an analytical model for the phase propagation in 1+1 dimensions, we prove that the phase waves generated in the bubble merging are reflected by the walls of the true vacuum cavity, giving rise to a long-lived oscillating state that delays the phase equilibration. The existence of such a state in the 3+1 dimensional model is then confirmed by numerical simulations, and the consequences for the formation of vortices in three-bubble collisions are considered.

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## I. INTRODUCTION

According to the standard model and its extensions, symmetry breaking phase transitions are expected to occur in the early universe. The mechanism by which these transitions may take place could be either by formation of bubbles of the new phase within the old one (i.e., first order phase transition) or by spinodal decomposition (i.e., second order phase transition). Although the verdict is still pretty much in the air as to which one would have actually taken place, in the particular case of the electroweak phase transition common opinion inclines more towards the first of the two possibilities. As this scenario would have it, bubbles of the new phase nucleated within the old one (the nucleation process being described by instanton methods as far as the WKB approximation remains valid [1]), and subsequently expanded and collided with each other until occupying basically all of the space available at the completion of the transition. The caveat introduced by the word “basically” is important though: in the process of bubble collision, the possibility arises that regions of the old phase become trapped within the new one, giving birth to topologically stable localized energy concentrations known as *topological defects* (for recent reviews see refs. [2]), much in the same way in which these structures are known to appear in condensed matter phase transitions. From a theoretical point of view, topological defects will appear whenever a symmetry group  $G$  is spontaneously broken to a smaller group  $H$  such that the resulting vacuum manifold  $M = G/H$  has a non-trivial topology: cosmic strings for instance, vortices in two space dimensions, will form whenever the first homotopy group of  $M$  is non-trivial, i.e.,  $\pi_1(M) \neq 1$ .

To see how this could happen in detail, let's consider the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi^* - V(\Phi) \quad (1)$$

for a complex field  $\Phi$ . If  $V$  is a potential of the type  $V = \frac{\lambda}{4} (|\Phi|^2 - \eta^2)^2$  and its parameters are functions of the temperature such that at high temperatures  $\Phi = 0$  is the only minimum of  $V$ , while at zero temperature all the  $|\Phi| = \eta$  values of the field correspond to different minima, then the structure of the vacuum manifold will be that of  $S^1$ . As  $\pi_1(S^1) \neq 1$ , cosmic strings should then be created in this model during the phase transition.

The way in which this would actually take place is via the Kibble mechanism [3]. The basic idea behind it is that when two regions in which the phase of the field takes different values encounter each other, the phase should interpolate between these two regions following a geodesic in the vacuum manifold. In the context of a first order phase transition, a possible scenario for vortex formation in two space dimensions would then look like this: three bubbles with respective phases of  $0$ ,  $2\pi/3$ , and  $4\pi/3$  collide simultaneously. According to the so-called geodesic rule then, if we walk from the first bubble to the second, then from the second to the third, and finally back from the third to the first again, the phase will have wound up by  $2\pi$  in our path, having traversed the whole of the vacuum manifold once along the way. Continuity of the field everywhere inside the region contained by our path demands then that the field be zero at some point inside of it, namely the vortex core. In the limit in which the bubbles extend to infinity, outwards from the center of collision, removal of this vortex would cost us an infinite amount of energy, since it would involve unwinding the field configuration over an infinite volume. The vortex is thus said to be topologically

stable. In three space dimensions, the resulting object would obviously be a string, rather than a vortex.

Clearly, there are other ways in which strings could be formed. Collisions of more than three bubbles could also lead to string formation, or, for instance, two of the bubbles could hit each other first, with the third one hitting only at some later time while the phase is still equilibrating within the other two. This event in particular will be far more likely than a simultaneous three way (or higher order) collision, and it is probably the dominating process by which strings are formed (especially if nucleation probabilities are as low as required for WKB methods to be valid). If this is the case, the importance of understanding the processes that take place when two bubbles collide, as well as the need to clarify the role of the third bubble and the conditions that should be met in order to get vortex formation, becomes clear.

Bubble collisions have been studied by a number of groups, most notably by Hawking *et al.* [4], Hindmarsh *et al.* [5], Srivastava [6] and more recently by Melfo and Perivolaropoulos [7], but important aspects of the question remain unclear so far. For instance only in [6] and [7] the posterior collision of a third bubble is treated (the first two references studied only two bubble collisions), and in none of the referred works did the authors concern themselves with the interaction between the bubble field and the surrounding plasma. From current work in the electroweak phase transition however, we can expect that in many cases such interaction will not be negligible (e.g, see Ref [8]). The basic underlying reason is relatively simple to understand: as the bubble wall sweeps through an specific point, the Higgs field  $\Phi$  acquires an expectation value, and the fields coupled to it acquire a mass. Thus, particles with not enough energy to acquire the corresponding mass inside the bubble will bounce off the wall (thus imparting negative momentum to it), while the rest will get through. Obviously, the faster the wall propagates the stronger this effect will be, since the momentum transfer in each collision will be larger, and thus a force proportional to the velocity with which the wall sweeps through the plasma appears.

The aim of this paper will thus be to explore two bubble collisions in an environment that damps the bubble motion to try and establish how phase diffusion and equilibration will proceed in this case, both analytically and numerically. The damping that we will consider here will only affect the modulus of the field though (which is equivalent to damping only the bubble wall motion), since, the phase being a Goldstone boson, we do not expect it to couple to the plasma as strongly as the modulus. We will find significant differences with the undamped case, the main new features being related to the possibility of the phase wave catching up with the bubble wall and bouncing off it. Once that is understood, we will study collisions with a third bubble and see under what circumstances we can expect to have vortex formation. All this will be done in the context of a global  $U(1)$  symmetry breaking model.

The paper is organized in three sections: in the first one we establish the general analytical model, in the second we study two bubble collision and phase bouncing in 1+1 and 2+1 dimensions (the 3+1 case should be analogous to the latter one), and finally the third is devoted to three bubble collisions in 2+1 dimensions.

## II. ANALYTICAL MODEL

Consider the Lagrangian (1) for a complex field  $\Phi$ . We will use the same form of potential that was used in [6,7], that is,

$$V = \lambda \left[ \frac{|\Phi|^2}{2} (|\Phi| - \eta)^2 - \frac{\epsilon}{3} \eta |\Phi|^3 \right]. \quad (2)$$

This is just a quartic potential with a minimum at  $|\Phi| = 0$  (the false vacuum), and a set of minima connected by a  $U(1)$  transformation (true vacuum) at  $|\Phi| = \rho_{tv} \equiv \frac{\eta}{4}(3 + \epsilon + \sqrt{(3 + \epsilon)^2 - 8})$ , towards which the false vacuum will decay via bubble nucleation. It is the dimensionless parameter  $\epsilon$  that is responsible for breaking the degeneracy between the true and the false vacua.

The equations of motion for this system are then

$$\partial_\mu \partial^\mu \Phi = -\partial V / \partial \Phi. \quad (3)$$

For the potential (2), approximate solutions of (3) exist for small values of  $\epsilon$ , the so-called thin wall regime [10], and are of the form

$$|\Phi| = \frac{\rho_{tv}}{2} \left[ 1 - \tanh \left( \frac{\sqrt{\lambda} \eta}{2} (\chi - R_0) \right) \right], \quad (4)$$

where  $R_0$  is the bubble radius at nucleation time and  $\chi^2 = |\vec{x}|^2 - t^2$ . The bubble then grows with increasingly fast speed and its walls quickly reach velocities of order 1. We are interested however in investigating a model with damped motion of the walls due to the interaction with the surrounding plasma. In order to model this effect, we will insert a frictional term for the modulus of the field in the equation of motion, namely

$$\partial_\mu \partial^\mu \Phi + \gamma |\dot{\Phi}| e^{i\theta} = -\partial V / \partial \Phi, \quad (5)$$

where  $|\dot{\Phi}| \equiv \partial|\Phi|/\partial t$ ,  $\theta$  is the phase of the field, and  $\gamma$  stands for the friction coefficient (which will as a matter of fact serve as parameter under which we will hide our lack of knowledge about the detailed interaction between the wall and the plasma). We have found that there are approximate analytical solutions to (5) corresponding to one bubble configurations that have reached their terminal velocity in the thin wall limit. Solutions will be found in a constructive way: first we will suppose that solutions for which the wall has the form of a traveling wave do exist to find out what their terminal velocity would be, and then use this expression for the velocity to obtain the detailed analytical form of such solutions in the thin wall limit. Comparison with numerical simulations will confirm the existence of such solutions. Writing  $\Phi$  in polar form:

$$\Phi = \rho e^{i\theta}, \quad (6)$$

the equation of motion (5) takes the form

$$\partial_\mu \partial^\mu \rho + \gamma \dot{\rho} = -\partial V(\rho)/\partial \rho + 2\rho \partial_\mu \theta \partial^\mu \theta, \quad (7a)$$

$$\partial_\mu \{\rho \partial^\mu \theta\} = 0. \quad (7b)$$

For a single bubble configuration we take  $\theta$  to be a constant (the phase of the bubble). Equation (7b) is then automatically satisfied and the equation left for the modulus of the field is

$$\partial_\mu \partial^\mu \rho + \gamma \dot{\rho} = -\partial V(\rho)/\partial \rho. \quad (8)$$

To get an approximate expression for the terminal velocity of a bubble under this equation of motion in the thin wall limit, note that because the wall thickness is much smaller than the radius of the bubble, we can go to 1+1 dimensions. Inserting then the ansatz  $\rho = \rho(x - x_0(t))$  leads to

$$(1 - \dot{x}_0^2) \rho'' + (\ddot{x}_0 + \gamma \dot{x}_0) \rho' = \partial V(\rho)/\partial \rho, \quad (9)$$

where  $\rho' \equiv \partial \rho / \partial x$ . We then get an effective equation for the wall motion simply by multiplying by  $\rho'$  and integrating over  $-\infty \leq x \leq +\infty$  to get rid of the extra degrees of freedom pertaining to the field, whence

$$(\ddot{x}_0 + \gamma \dot{x}_0) \left( \int_{-\infty}^{+\infty} \rho'^2 dx \right) = \int_{-\infty}^{+\infty} V' dx = \Delta V, \quad (10)$$

where  $\Delta V$  is just the potential energy difference between the false and the true vacuum phases, and we have used the fact that derivatives go to zero far from the origin. The solution to (10) for the initial conditions  $x_0(t=0) = R_0$ ,  $\dot{x}_0(t=0) = 0$  is

$$x_0(t) = \frac{1}{\gamma} \alpha t + \frac{\alpha}{\gamma^2} (e^{-\gamma t} - 1) + R_0 \quad (11)$$

where  $\alpha \equiv \Delta V / (\int \rho'^2 dx)$ . Thus, for values of  $t \gg \gamma^{-1}$  the bubble walls will have reached their terminal velocity

$$v_{ter} = \frac{\Delta V}{\gamma (\int \rho'^2 dx)}. \quad (12)$$

Although obtained in a slightly different way, this expression coincides with what Heckler found in [11]. Typical estimates for the terminal velocities of thin wall bubbles in the electroweak phase transition give  $v_{ter} \sim 0.1$  [8] (although in recent work higher values have been found, see [9]).

To get an approximate expression for  $\rho$  valid within this regime, it suffices to rewrite (8) with a  $\rho = \rho(r - r_0(t))$  ansatz, where  $r$  is the usual radial coordinate. Using  $\ddot{r}_0 = 0$ ,  $\dot{r}_0 \approx v_{ter}$ , we get

$$(1 - v_{ter}^2) \frac{\partial^2 \rho}{\partial r^2} + \left( \frac{2}{r} + \gamma v_{ter} \right) \frac{\partial \rho}{\partial r} = \partial V(\rho)/\partial \rho, \quad (13)$$

$r$  being the radial coordinate in 3 spatial dimensions. According to (12) however the terminal velocity goes roughly like

$$v_{ter} \approx \frac{\Delta V}{\gamma \left( \frac{\rho_{tv}^2}{\delta_m^2} \delta_m \right)} = \frac{\Delta V \delta_m}{\gamma \rho_{tv}^2}, \quad (14)$$

where  $\delta_m$  is the bubble wall thickness and  $\rho_{tv}$  is the true vacuum value of the field. That is, at the values of  $r$  for which the first derivative of the field is important ( $r \sim R$  for a thin wall bubble), we have  $R\gamma v_{ter} \sim \delta_m \ll 1$ , and the second summand in the parenthesis is negligible when compared to the first. Furthermore, since the radius of thin-walled bubbles is very large, we can also neglect the term  $(2/r) \partial \rho / \partial r$ —this is just the standard thin wall approximation. Thus, we are left with

$$(1 - v_{ter}^2) \frac{\partial^2 \rho}{\partial r^2} = \partial V(\rho) / \partial \rho, \quad (15)$$

whose solutions for the potential written in (2) will be

$$\rho = \frac{\rho_{tv}}{2} \left[ 1 - \tanh \left( \frac{\sqrt{\lambda} \eta}{2} \frac{(r - v_{ter} t - R_0)}{\sqrt{1 - v_{ter}^2}} \right) \right], \quad (16)$$

which is simply a Lorentz-contracted, moving domain wall.

We have followed closely the analytical study with numerical simulations of the processes. This was done in two steps (for details see Ref [7]): first we find a numerical solution of the Euclidean equations of motion for the field, whose analytic continuation into Minkowski space will give us the shape of the instanton solution immediately after the tunneling has taken place. The initial bubbles thus found were then placed in a two dimensional lattice, and evolved according to (5) with a modified leapfrog method and reflecting boundary conditions. In the following sections, we will be comparing the analytical results with these simulations.

### III. TWO BUBBLE COLLISIONS: PHASE PROPAGATION AND BOUNCING

We will now try to take a look at the events that take place when two damped bubbles collide. To do so, we will organize the section into three parts. The first one will deal with the configuration and evolution of the phase from the time when the bubbles are still very far away from each other to the point when they have completed the collision process. The second one takes on from there to study the propagation inside the bubbles of the phase waves that originate at that time, and stretches out until the moment when these waves catch up with the walls at the other end of the system. Finally, the third part studies the interaction between the phase waves and the bubble walls, finding how the former will bounce off the latter and propagate again into the interior of the bubbles, which will thus behave almost as a resonant cavity.

#### A. Initial phase configuration

Within the WKB approximation the bubble nucleation rate is small, and therefore typical bubbles are nucleated a long distance apart from each other. The approximate solution for a system of two bubbles is then simply the sum of the two independent bubble solutions,

$$\Phi(bubble1 + bubble2) \equiv \rho e^{i\theta} \simeq \Phi(bubble1) + \Phi(bubble2) \equiv \rho_1 e^{i\theta_1} + \rho_2 e^{i\theta_2}, \quad (17)$$

to exponential accuracy. From this it is easy to compute how the phase interpolates between the two bubbles

$$\tan \theta = \frac{\rho_1 \tan \theta_1 + \rho_2 \tan \theta_2}{\rho_1 + \rho_2}, \quad (18)$$

which has a kink like shape centered at the origin. One could worry however that, as the bubbles move towards each other, phase waves could be generated in the central region that would subsequently propagate over the exponentially small (but finite) background of the modulus so that (18) would no longer adequately represent the combined phase of the two bubbles in motion, even if (17) still provided a good ansatz for the combined modulus. To check whether this is indeed the case we can solve the equation of motion for the phase (7b) using the modulus given by (17). In a 1+1 dimensional approximation, near  $|x| = 0$  and far away from the centers of each bubble we can take their moduli to be (we will write simply  $v$  for  $v_{ter}$  now)

$$\begin{aligned} \rho_1 &\simeq \rho_{tv} e^{(x+vt-x_0)/\delta_m}, \\ \rho_2 &\simeq \rho_{tv} e^{-(x-vt+x_0)/\delta_m}, \end{aligned} \quad (19)$$

where we have taken the centers of the two bubbles to be initially situated at  $\pm x_0$ . Then (17) yields for the combined modulus

$$\rho^2 \simeq \rho_{tv}^2 e^{2(vt-x_0)/\delta_m} \{2 \cosh(2x/\delta_m) + 2 \cos(\theta_1 - \theta_2)\}. \quad (20)$$

We can now use this in the equation of motion for the phase (7b), to get

$$\ddot{\theta} - \theta'' + \frac{2v}{\delta_m} \dot{\theta} - \frac{2\delta_m^{-1} \sinh(2x/\delta_m)}{\cosh(2x/\delta_m) + \cos(\theta_1 - \theta_2)} \theta' = 0. \quad (21)$$

Note that for an initial phase difference  $\theta_1 - \theta_2 = \pi$  the modulus of the field (20) is zero at the midpoint between the bubbles. In this case then the denominator in the last term of (21) goes to zero, which means that the phase has the shape of a step function as we go over the origin, switching discontinuously from  $\theta_1$  to  $\theta_2$ . On the other extreme, if the initial phase difference is zero there is of course no dynamics to it. We will then focus in an intermediate situation, and find solutions to (21) for a phase difference of  $\pi/2$ . Using the ansatz  $\theta = T(t)X(x)$ , we can separate variables to obtain:

$$\ddot{T} + \frac{2v}{\delta_m} \dot{T} + k^2 T = 0, \quad (22a)$$

$$X'' + \frac{2}{\delta_m} \tanh\left(\frac{2x}{\delta_m}\right) X' + k^2 X = 0, \quad (22b)$$

$k$  being the separation constant. The equation for  $T$  is obviously that of a damped oscillator. For  $k = 0$ , a  $T = \text{const.}$  solution exists consistent with our boundary condition that  $\theta$  goes to  $\theta_{1,2}$  as  $x \rightarrow \pm\infty$  at all times, while, for  $k \neq 0$ , solutions with temporal dependence on the phase will die away on time scales of the order of  $\delta_m/v$  (as a matter of fact these solutions

would seem to have been artificially introduced by the switching on of the interaction between the bubbles). To exponential accuracy then, the phase will rapidly settle in a stationary state that will interpolate between its two values at infinity. The solution to (22b) for  $k = 0$  interpolating from say  $\theta_1 = 0$  at  $x \rightarrow -\infty$  to  $\theta_2 = \pi/2$  at  $x \rightarrow +\infty$  will thus give us its complete behavior:

$$\theta = \frac{1}{2} \arcsin(\tanh(\frac{2x}{\delta_m})) + \frac{\pi}{4}. \quad (23)$$

Thus, the near false vacuum state around the bubbles behaves as a rather effective insulating material as far as conduction of phase waves is concerned, and we should not expect any significant propagation of the phase until the bubbles run into one another. It may also be worth noting how the phase interpolates from one of its values to the other by means of a wall-like structure similar to walls created by the modulus of the field. The thickness of this phase wall,  $\delta_p$ , seems to be closely related to the thickness of the bubble wall  $\delta_m$ , although this relation will obviously change as the phase difference between the bubbles change.

Once the bubbles collide and their walls begin to merge, the phase of each bubble will start propagating into the other one at the speed of light. While this merging is taking place, its net effect will be to widen the phase wall, so that its final thickness will be *at least* twice the thickness of the bubble walls for relativistic bubbles.

Figure 1 shows the results of simulations of this process. We have plotted the phase and the bubble walls for a 1-dimensional, 2 bubble configuration with an initial phase difference of  $\pi/2$ , for three different times of the evolution: when they are about to start colliding, at the middle of the merging process, and at the point of its completion respectively. We can clearly see how the phase wall smoothens out and thickens while this takes place.

## B. Phase propagation inside the bubbles

Once the bubble walls have completed their merging, the phase is free to propagate in the resulting single cavity. Since, especially in the thin wall case, the modulus of the field inside the cavity remains essentially constant and equal to its true vacuum value, the equation of motion for the phase simplifies to a wave equation

$$\partial_\mu \partial^\mu \theta = 0. \quad (24)$$

Kibble and Vilenkin [12] have studied the process of wave propagation for the Abelian gauge model without dissipation. In the case they studied the bubbles move essentially at the speed of light, and the problem has a high degree of symmetry: if the axes are chosen so that the bubbles nucleate along the  $z$  axis at say  $(0, 0, 0, \pm R)$ , the whole bubble collision process will be invariant under the 3-dimensional Lorentz group  $SO(1,2)$  in the  $(t, x, y)$  subspace. The bubble collision occurs then along the surface  $z = 0$ ,  $t^2 - x^2 - y^2 = R^2$ , and for any point in that surface there will be a frame of reference in which that is the point of first contact. Thus, symmetry dictates that  $\theta$  be a function

$$\theta = \theta(\tau, z), \quad (25)$$

where  $\tau^2 = t^2 - x^2 - y^2$ . In our case, the damped motion of the walls obviously breaks Lorentz invariance as far as the bubbles motion and collision is concerned. The points at



which the two bubbles connect to each other in the  $z = 0$  plane however will still move at a very high speed (in fact much higher than the terminal velocity of the walls), and, since the phase is not affected by the damping, SO(1,2) Lorentz invariance should still be a good approximation in our case. Inserting the ansatz (25) for  $\theta$  in (24) yields then the equation for the phase

$$\partial_\tau^2 \theta + \frac{2}{\tau} \partial_\tau \theta - \partial_z \theta = 0. \quad (26)$$

As initial setup we will assume that the bubbles have been nucleated at  $\pm R$  in the  $z$  axis while the collision takes place at  $t = 0$ . For the initial conditions we will take the thickness of the phase wall to be negligible when compared to the other scales of the problem, that is,

$$\theta|_{\tau=0} = \theta_0 \varepsilon(z), \quad \partial_\tau \theta|_{\tau=0} = 0, \quad (27)$$

where  $\varepsilon$  is the step function. Equation (26) can then be solved to get

$$\theta = \begin{cases} \theta_0 & \text{for } z > \sqrt{t^2 - x^2 - y^2}, \\ \theta_0 \frac{z}{\sqrt{t^2 - x^2 - y^2}} & \text{for } |z| \leq \sqrt{t^2 - x^2 - y^2}, \\ -\theta_0 & \text{for } -z > \sqrt{t^2 - x^2 - y^2}. \end{cases} \quad (28)$$

This approximation will obviously work better for bubbles that move with relatively high terminal velocities. Firstly of course because the Lorentz invariance approximation will be more adequate, but also because faster wall motion leads to smaller phase wall thicknesses, as seen at the end of the last subsection, so that (27) is a better approximation for the phase wall. It will also clearly work better in the central region of the bubble than close to its walls, where the interaction between these and the phase wave, as well as the detailed way in which the connecting circumference between the two bubbles expands, may be of importance.

To see how well this approximation holds we can compare it with numerical simulations, which will however be performed in  $2 + 1$  dimensions. The solution to the  $2 + 1$  version of (26) is easily found

$$\theta = \begin{cases} \theta_0 & \text{for } z > \sqrt{t^2 - x^2}, \\ \frac{2\theta_0}{\pi} \arcsin\left(\frac{z}{\sqrt{t^2 - x^2}}\right) & \text{for } |z| \leq \sqrt{t^2 - x^2}, \\ -\theta_0 & \text{for } -z > \sqrt{t^2 - x^2}, \end{cases} \quad (29)$$

and has general features similar to those of the  $3 + 1$  case. Figure 2a shows a couple of snapshots of the phase wave profile along the  $z$  axis at different times after the collision for thin wall bubbles with relatively low friction. As we can see, our results give a rather accurate picture of how the phase evolves in this case. Figure 2b shows the same situation for the case in which the bubbles move under high friction (i.e., low terminal velocity). As we can see, here the phase wall thickness at the time at which the bubbles finish merging is rather large, and therefore the step function approximation in (27) fails to correctly represent the initial state of the phase. We should note however that the phase propagation seems to proceed along similar lines as before, so that if we were to solve (26) with a smoother ansatz for  $\theta(\tau = 0)$  we should again reproduce the observed phase behavior.

### C. Interaction between the phase wave and the bubble walls: phase bouncing

The most salient feature of the problem of bubble collision in the damped regime will be the possibility of interaction between the phase waves that propagate inside the merged bubbles, and the bubble walls. Such a situation was of course never encountered in the case of walls moving in vacuum and reaching asymptotically the speed of light.

We can roughly anticipate the outcome of this interaction on physical grounds: since the phase is massless inside the bubbles, but massive in the near false vacuum outside them, it should follow that only the contributing modes to the phase wave which have sufficient energy to acquire the required mass will be able to go through the wall at all, while all the others must bounce off it. Whether these modes will exist at all in the phase wave, and if they do in what proportion, will then determine how the phase wave will behave after the bouncing.

We will first study the interaction between the phase wave and the bubble wall both analytically and numerically in the simpler  $1 + 1$  dimensional scenario, and then try to extrapolate our conclusions to the  $2 + 1$  case and confront them with more numerical simulations there.

In  $1 + 1$  dimensions, our damped thin wall solution looks like

$$\rho = \frac{\rho_{tv}}{2} \left[ 1 - \tanh \left( \frac{\sqrt{\lambda}\eta}{2} \frac{(x - v_{ter}t - x_0)}{\sqrt{1 - v_{ter}^2}} \right) \right]. \quad (30)$$

Boosting to a frame of reference that moves along with the wall and has its origin at its center leads then to

$$\rho = \frac{\rho_{tv}}{2} \left[ 1 - \tanh \left( \frac{\sqrt{\lambda}\eta}{2} x \right) \right]. \quad (31)$$

We will be interested in the situation in which phase waves approach the wall from  $x \rightarrow -\infty, t \rightarrow -\infty$  and look at the asymptotic outcome for  $t \rightarrow +\infty$ , assuming that the incoming phase waves carry a very small amount of energy as compared to that stored in the wall so that we can neglect any back reaction on it due to the collision. In these conditions we can then take the shape of the wall as being essentially fixed by (31). The equation of motion for the phase simplifies to

$$\ddot{\theta} - \theta'' - 2\frac{\rho'}{\rho}\theta' = 0, \quad (32)$$

or, doing a Fourier transform in  $t$ ,

$$\theta'' + 2\frac{\rho'}{\rho}\theta' + \omega^2\theta = 0. \quad (33)$$

Using then (31) for  $\rho$  as advertised and performing the change of independent variable

$$y = \frac{1}{1 + e^{-\sqrt{\lambda}\eta x}} \quad (34)$$

puts the phase equation in the form

$$y(1-y)\theta'' + (1-4y)\theta' + \frac{\omega^2}{\lambda\eta^2 y(1-y)}\theta = 0, \quad (35)$$

where now the prime indicates derivatives with respect to  $y$ . Equation (35) is a form of Papperitz equation with regular singular points at  $0, 1, +\infty$ , (see Ref. [12]) and consequently its solutions will be given by hypergeometric functions.

To learn about the fate of an incoming wave we will look at the two independent solutions of (35) around  $y \rightarrow 1$  (i.e.,  $x \rightarrow +\infty$ ) and select the one that represents an outgoing wave. In terms of  $x$ , these outgoing wave solutions are found to be ( $B$  being an arbitrary constant)

$$\theta_{x \rightarrow +\infty} \simeq \begin{cases} (-1)^{\tilde{\omega}-1} B e^{x/\delta_m} e^{ix\tilde{\omega}/\delta_m}, & |\omega|\delta_m > 1 \\ (-1)^{\tilde{\omega}-1} B e^{x/\delta_m} e^{x\tilde{\omega}/\delta_m}, & |\omega|\delta_m < 1 \end{cases} \quad (36)$$

with  $\tilde{\omega} = \sqrt{1 - \omega^2/(\lambda\eta^2)}$ , and where for notational convenience we have used  $\delta_m \equiv (\sqrt{\lambda}\eta)^{-1}$  for the bubble wall thickness. (Note that the exponential growth of the solutions would apparently yield a diverging energy density, specially for the case  $|\omega|\delta_m < 1$ , more on this later).

Using now the connection formulas for the hypergeometric functions, we can express  $\theta$  in terms of incoming and outgoing waves around  $x \rightarrow -\infty$ . We get

$$\theta_{x \rightarrow -\infty} \simeq (-1)^{\tilde{\omega}-1} B [\Gamma_1 e^{i\omega x} + \Gamma_2 e^{-i\omega x}]. \quad (37)$$

where  $\Gamma_1, \Gamma_2$  are combinations of gamma functions. Demanding that the incoming wave has unit amplitude fixes the constant  $B$  and determines the reflection and transmission coefficients. We obtain as final expressions for  $\theta$

$$\theta_{x \rightarrow -\infty} \simeq e^{i\omega x} + \frac{\Gamma_2}{\Gamma_1} e^{-i\omega x}, \quad (38a)$$

$$\theta_{x \rightarrow +\infty} \simeq \begin{cases} (\Gamma_1)^{-1} e^{x/\delta_m} e^{ix\tilde{\omega}/\delta_m}, & |\omega|\delta_m > 1 \\ (\Gamma_1)^{-1} e^{x/\delta_m} e^{x\tilde{\omega}/\delta_m}, & |\omega|\delta_m < 1 \end{cases}. \quad (38b)$$

It only rests now to write down the precise form of the gammas and analyze their behavior. We get

$$\frac{\Gamma_2}{\Gamma_1} = \frac{\Gamma(1 + 2i\omega\delta_m)\Gamma(2 - \tilde{\omega} - i\omega\delta_m)\Gamma(-1 - \tilde{\omega} - i\omega\delta_m)}{\Gamma(1 - 2i\omega\delta_m)\Gamma(2 - \tilde{\omega} + i\omega\delta_m)\Gamma(-1 - \tilde{\omega} + i\omega\delta_m)} \quad (39)$$

Thus, for as long as  $\tilde{\omega}$  remains real (i.e., for as long as  $|\omega|\delta_m < 1$ ), numerator and denominator in (39) will be complex conjugate of each other, and the reflection coefficient will be

$$R = \left| \frac{\Gamma_2}{\Gamma_1} \right|^2 = 1. \quad (40)$$

When  $|\omega|\delta_m > 1$ ,  $\tilde{\omega} \rightarrow i\tilde{\omega}$  however, and  $R$  is

$$R = \frac{\sinh^2\left(\pi\left(\omega\delta_m - \tilde{\omega}\right)\right)}{\sinh^2\left(\pi\left(\omega\delta_m + \tilde{\omega}\right)\right)}, \quad (41)$$

and we see that  $R \rightarrow 0$  for  $\omega\delta_m \gg 1$ . Thus incident waves with wavelength  $1/\omega$  larger than the wall thickness will be completely reflected by the wall, whereas waves with wavelengths shorter than the wall thickness will be partially reflected and partially transmitted, the reflected part going to zero quickly as the wavelength decreases. Of course these conclusions will have to be somewhat modified to describe the process from the point of view of a static observer at the origin. For such a reference frame, a boost with velocity  $-v_{ter}$  yields for the new frequencies  $\omega'$  of the incident and reflected waves

$$\begin{aligned} \omega'_{in} &= \beta\omega(1 + v_{ter}), \\ \omega'_{ref} &= \beta\omega(1 - v_{ter}) = \omega'_{in} \frac{1 - v_{ter}}{1 + v_{ter}}, \end{aligned} \quad (42)$$

where  $\beta \equiv 1/\sqrt{1 - v_{ter}^2}$ . Thus, for a static observer the condition  $\omega\delta_m < 1$  is transformed into

$$\omega'_{in}\delta_m < \beta(1 + v_{ter}), \quad (43)$$

while the reflected wave will present a Doppler shift relative to the incoming wave due to it having bounced off a moving wall.

Figures 3a, 3b show an incident wave train of wavelength roughly five times that of the wall (in the static frame) being completely reflected. The reflected wave has the same amplitude as the incident one, but its frequency has decreased due to the Doppler shift. Figures 4a, 4b then show how a wave train with wavelength about half the wall thickness is partially reflected. Both cases present exponential growth of the phase outside the bubble (remember that, modulo  $2\pi$ , the maximum phase difference that one can have is  $\pi$ ). This seems to fit the general behavior (36). Note however that an exponential growth of the phase with  $x$  does not necessarily lead to an equivalent growth of the energy stored by the phase gradient, since this goes like  $\eta^2(\partial\theta/\partial x)^2$  and the modulus is decaying exponentially. Thus, in the expression for the case  $\omega\delta_m > 1$  the energy in the phase gradient will tend to a constant for instance. If the behavior of the phase outside the bubble were to be like the one given by the  $\omega\delta_m < 1$  regime in the same equation however, the phase gradient energy *would* diverge exponentially with  $x$ . Any of these two behaviours however will end up in the breakdown of our approximation that the wall is unaffected by the phase wave propagation. Once the phase wave gets through to the false vacuum, the modulus of the field stops behaving like a transparent medium as we saw in section III.1. In those conditions, and since the modulus decays exponentially, no matter how small was the energy carried initially by the phase wave we will always get to a point in which it will be of the same order than the energy of the modulus. At that point, backreaction on the modulus is no longer negligible, and a proper analysis of the problem would require solving the two coupled equations. The situation is depicted in Figures 3c, 4c, where a blow up shows how the wall

develops what appear as small bumps in its decaying region. Figures 3d, 4d then, where the energy of the modulus and of the phase are plotted, show how these bumps appear at the points where the energy of the phase is getting larger than that of the modulus. The result of this backreaction seems to be that the modulus absorbs the excess energy in the phase, whereas the phase will still propagate forward in some exponential way. We have not carried out a detailed analysis of this part of the problem however, since it does not directly affect the behavior of the phase inside the bubbles in any relevant way.

To sum up, we can expect that waves with wavelengths (roughly) larger than the bubble thickness will be totally reflected by the wall, the rest being partially transmitted and partially reflected by it. It is then easy to see what will happen to the phase wave in a (1+1 dimensional) two bubble collision: as we saw in section III.1, the phase wall thickness at the end of the collision will be *at least* of the order of twice the bubble wall thickness, and more likely larger than that. It seems clear then that basically all of the Fourier components of the phase wave will have wavelengths that will fall into the total reflection regime, and thus the whole phase wave itself will simply be reflected by the wall, while some form of exponential behaviour propagates to the outside of the bubbles. Let us imagine for the sake of clarity that, at collision time, bubble 1 had a phase  $\theta_1$  and bubble two  $\theta_2 > \theta_1$ , with  $\theta_2 - \theta_1 = \Delta\theta$ . If the shape of the phase wall at the completion of the collision was  $f(x)$  then, after it, we will have phase waves with shape  $f(x-t)/2$ ,  $f(x+t)/2$  propagating into each bubble, carrying a phase difference  $-\Delta\theta/2$  into bubble 2 and  $\Delta\theta/2$  into bubble 1. After these waves have bounced off the bubble walls and propagated back into the interior again, the phase of each bubble will however be, for bubble 2,  $\theta_2 - 2\Delta\theta/2 = \theta_1$ , and for bubble 1,  $\theta_1 + 2\Delta\theta/2 = \theta_2$ . The phases of the bubbles will thus have switched. The whole process is depicted in Figure 5, where the referred sequence has been plotted from a simulation. In Fig. 5a, the walls of the two bubbles are just about to finish their merging (continuous line), and the shape of the phase wall at that time is shown (dashed line). The bubble to the right plays the role of bubble 2 above, having  $\theta_2 > \theta_1$ . The following pictures show how the two phase waves propagate into the bubbles (Fig. 5b), and back after bouncing (Fig. 5c). As expected, after reflection each phase wave still carries a phase of  $\pm\Delta\theta/2$ , for bubbles 1 and 2 respectively. Finally, in Fig. 5d the two phase waves meet again. The phase polarity of the system has been completely inverted.

If this mechanism had no energy losses, the two bubbles would then behave as a sort of resonator in which the phase of each bubble would continuously oscillate between  $\theta_1$  and  $\theta_2$ , their respective polarities always switched. Since the bubble walls are moving though, the phase waves present the expected Doppler shift after bouncing off the walls as was seen before (which can also be clearly seen in Figs. 5c, d). For non-relativistic terminal velocities (i.e., for  $v_{ter} \ll 1$ ), the magnitude of this shift is from (42) (we drop the primes in the notation for the frequencies here)

$$\omega_{ref} \simeq \omega_{in}(1 - 2v_{ter}), \quad (44)$$

or what is the same, after  $n$  reflections, the resulting phase wave will have a frequency

$$\omega_{nref} \simeq \omega_{in}(1 - 2v_{ter})^n. \quad (45)$$

The oscillation process will effectively die off when  $\omega_{nref} \sim (2\mathcal{R})^{-1}$ ,  $\mathcal{R}$  being the (approximate) radius of the single true vacuum cavity after  $n$  reflections have taken place. If  $\omega_{in}^{-1}$  is

very small when compared to  $\mathcal{R}$ , and neglecting higher orders in  $v_{ter}$ ,  $\omega_{nref}$  becomes then of the order of the radius after

$$n \simeq \frac{1}{2v_{ter}} \quad (46)$$

oscillations. On the other hand, it is not difficult to see that the time needed to complete each oscillation grows like

$$t_n = 4R_0 \left( \frac{1}{1 - v_{ter}} \right)^n, \quad (47)$$

where  $R_0$  is the radius of the bubbles at collision time (keeping in mind that it will take only half  $t_1$  to reach the walls for the first time though). Thus, the relaxation time until the oscillations die off will be

$$\tau_{osc} \simeq \frac{2R_0}{1 - v_{ter}} + \sum_{n=2}^{n \simeq 1/2v_{ter}} 4R_0 \left( \frac{1}{1 - v_{ter}} \right)^n. \quad (48)$$

For a terminal velocity around 0.1, (46) gives us five oscillations until equilibration, for a total oscillating time  $\tau_{osc} \sim 24R_0$  from (48). We see therefore that the relaxation time for the system can in general be quite significant.

#### IV. THREE BUBBLE COLLISIONS: VORTEX FORMATION

We wish now to try to generalize the results found for phase wave bouncing in 1+1 dimensions to 2+1 dimensions, since our final concern is to find whether these effects can affect the formation of vortices. Finding an exact solution for the phase propagation and interaction in 2+1 dimensions is an extremely involved problem, but the results in one spatial dimensions can certainly be used as a guide, and numerical simulations can provide the rest.

Note first that wave propagation in two or three spatial dimensions differs qualitatively from the one dimensional case. Whereas in one dimension we have two phase waves that propagate inside the bubbles carrying half of the phase difference each, in two or more dimensions we have two wavefronts with amplitudes that decrease in time and a region that continuously interpolates between them (see the previous section). In Figure 6 we have plotted magnitude and phase contours in a two-bubble collision, as well as the phase at each point represented by an arrow. We can see how the phase interpolates between the two bubbles at merging (6a), and then starts to propagate. During the initial stages of propagation, the points of contact of the two bubbles move in the direction perpendicular to the  $z$  axis at superluminal speed, so that the phase waves cannot reach them. Propagation is similar in these first stages to the undamped case: the region of interpolated phase is just a semicircle (6b). Soon however the contact points will slow down to asymptotically reach the terminal velocity. When the phase waves reach the walls, the shape of the interpolating region is affected by the interaction, as can be seen in 6c and d. We come then to the first consequence of the damped motion of the walls. The fact that the phase waves are able to catch up with the bubble walls means that, at this point, a region of smoothly interpolated (i.e., nearly homogeneous) phase extends up to the boundaries of the system. At this stage

then it is easy for a third bubble of the right size and position to collide with the merged two-bubble system *inside* the region where the phase is nearly homogeneous. Hence, no vortex will be formed. The situation is depicted in Fig. 7, where a third bubble collides with the two bubble system while the processes seen in Figure 6 are taking place. In 7a and b, the third bubble collides shortly after the phase starts interpolating, so that a vortex is created. In 7c and d however the nucleation of the third bubble occurs a little later. When it finally reaches the two bubble system, the phase has already interpolated beyond the collision points and no vortex is formed. Notice that the initial phases of the bubbles in these figure are not exactly  $\pi/6$ ,  $5\pi/6$  and  $3\pi/2$ , but the two initial bubbles have slightly smaller values of the phases, so that the resulting interpolated phase after merging is slightly smaller than  $\pi/2$ . This was done in order to avoid having the third bubble colliding with a region of *exactly opposite* phase. As was shown in Ref. [7], such a collision produces a long-lived “domain wall” state between the bubbles that delays the merging significantly. We will come back to this point after we discuss the further evolution of the system below, for the time being we just remark that the phase distribution is responsible for the slight asymmetry in the vortex’s position.

The situation depicted in Fig. 7 could represent a rather efficient mechanism for the suppression of vortex formation, if it were not for the interaction with the walls. From the 1+1 case, one expects the phase wave to bounce back, thus spoiling its homogenization. How does the bouncing look like in 2+1 dimension? In two spatial dimensions, at any point that the phase wave meets the wall, its propagation vector will have one component in the direction normal to the wall and another one tangential to it. Only the component in the normal direction will see the wall and bounce off it however, while the other one will continue to propagate freely. Thus, although in general the interaction between the wall and the phase wave will be a complicated superposition of these two processes, we can expect that after some time, in the region close the walls the phase will predominantly propagate tangentially to them, the rest of it having bounced towards the center. The effects of this bouncing can already be seen in Figures 6c and d: it is the reflection of the normal modes that causes the change of shape in the wavefront.

After this we only have to wait for the central region of the phase, propagating along the  $z$  axis, to get to the end of the bubble and bounce off the wall. Fig. 8 is a continuation of the same simulation started in Fig. 6. At the stage depicted by 8a, the phase wave has reached the end of the bubble. Therefore, at that point virtually all the phase propagating normal to the wall has bounced towards the center, and tangential propagation dominates close to the wall. In Figure 8b we see the subsequent evolution of the system. While the central region of the phase, propagating along the  $z$  axis, collides head on and bounces off the end of the bubble as expected, the tangential components cross each other at that region and start propagating in the opposite direction, back towards the center of the collision. Thus, the combination of these two phenomena brings about an inversion of the phase similar to that found in 1+1 dimensions. Finally, a third bubble collides with the system in 8c, *after* the bouncing has taken place. Because of phase inversion, we get an antivortex formed in 8d.

Note that if we were to wait for the collision to happen until yet another bounce had taken place (for a total of two bounces), the resulting defect would again be a vortex. Three bounces and an antivortex again, and so on. To get a total suppression of the vortex

formation probability, therefore, one would have to delay the collision of the third bubble until the system has relaxed. As we have seen, the 1+1-dimensional model predicts a large value for the relaxation time. Simulations show that for  $v_{ter} \sim 0.1$ , vortices can still be formed in collisions occurring after the merged system has gone through five oscillations, our predicted time for equilibration. So the two-bubble system will have to remain in “isolation” for a very long time to be able to acquire an homogeneous distribution of phase, a very unlikely situation. Our results seem to indicate then that an analysis of the vortex formation probability will have to take into account multiple bubble collisions, such as for example the one carried out in Ref. [13]

The situation is of course further complicated if we consider different initial positions for the colliding bubbles. We have taken them to be situated so that the latest bubble is centered in the plane of collision of the other two, but it is easy to see that any shift in position will affect the formation of vortices in a non trivial way, as it does any change in the nucleation time. We conclude with some more words about the case depicted in Fig. 7c and d. Had we set the three phases equally distributed, the delay caused in the merging due to the formation of a metastable wall between the region of opposite phases will have given enough time for the bouncing to take place, and an antivortex to be formed. Simulations show that for the positions of the bubbles considered here, a defect is *always* formed in the particular (and less likely) case of equally distributed initial phases, unless the phase oscillations have completely relaxed.

## V. CONCLUSIONS

We have analyzed bubble motion and interactions in a plasma, simulating its damping effect by a friction term in the equations of motion. We have found that there exists an exact solution for the damped motion of the bubble in the thin-wall regime, representing a bubble propagating with a terminal speed roughly equal to the inverse of the friction coefficient.

An analytical study of the collision of two bubbles, and the subsequent process of phase interpolation, was then performed. First, the shape of the “phase wall” that interpolates between the bubbles was found, and it was shown that the vacuum state outside the bubbles is a very efficient “insulator”, so that one does not expect the phase to propagate in the false vacuum. The initial thickness of the phase wall was shown to be at least twice that of the bubble’s walls.

Once the initial set up for the collision was thus determined, the equations for the phase propagation were solved, and the interaction with the walls determined, using the previously found solution for the wall’s motion. It was shown that phase waves generated in a collision have wavelengths such that they are always reflected by the walls. The result is that an oscillating state is formed, in which the bubble’s walls act almost as a resonant cavity. Since the true vacuum cavity expands, though, these oscillations are damped and eventually die off. For terminal velocities typical of those expected in an electroweak phase transition, the relaxation time is estimated to be around  $24R$ , with  $R$  the radius of the bubbles at collision.

Using numerical simulations, these results were shown to hold in 2+1 dimensions. The reflection of the phase wave goes along similar lines than the one-dimensional case. The formation of vortices was shown to be affected by the damping motion, basically due to two facts: first, that the interpolated phase region can reach the boundaries of the system,



where a reflection of the phase waves off the bubble walls will take place; and second that the long-lived oscillatory state thus produced delays significantly the homogenization of the phase. The main effect of this oscillatory state will have on vortex formation processes will be that it will become possible, for the same set of three bubbles, to form a vortex, and antivortex or no defect at all depending on the precise timing of the last collision.

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## FIGURES

FIG. 1. Modulus of the field (a) and phase (b) of a system of two bubbles, along the axis that joins their centers, for three moments of the collision process. Continuous line is for  $t = 16$ , dotted for  $t = 19$ , dashed for  $t = 20$  (where radial and time coordinates are given in units of the field's mass, and the modulus is normalized with the symmetry-breaking scale). Bubbles are nucleated at  $t = 0$ , and  $\gamma = 3$ . Here and in the following graphs,  $\epsilon = 0.8$ .

FIG. 2. Phase propagation inside the bubbles. In (a), the friction coefficient  $\gamma$  is 3, in (b) is 10. The analytical solution is shown for different times with a continuous line. Results of the simulation of the phase propagation are shown with triangles (for  $t = t_c + 1$ ,  $t_c =$  collision time) and circles (for  $t = t_c + 7$ ).

FIG. 3. Interaction of a phase wave with the bubble wall, for a phase wavelength approximately equal to five times the width of the wall. Arrows indicate the direction of propagation of bubble and phase. In (a), (b) and (c), the field's phase (dashed line) and magnitude (continuous line) are plotted for different times as indicated. In (d), the energy in modulus (continuous line) and phase (dashed line) are compared.

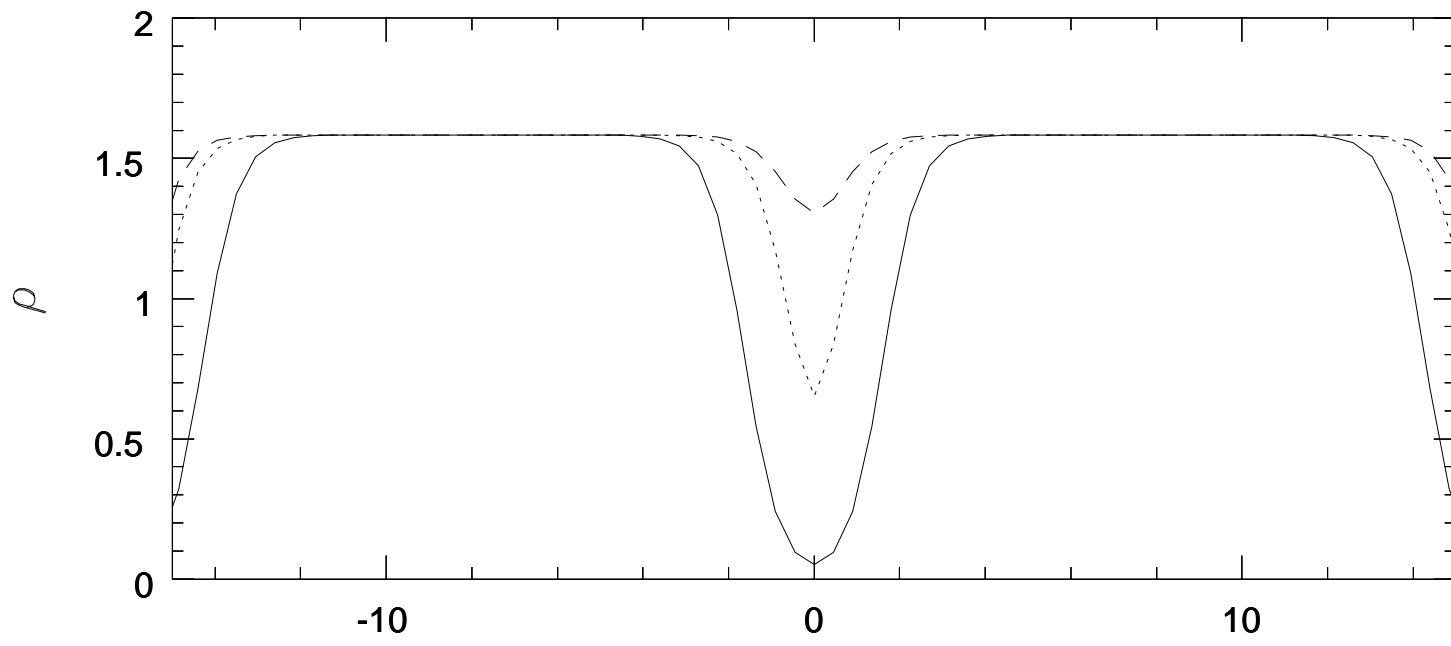
FIG. 4. Same as Fig. 3, where now the phase wave has a wavelength of approximately half the wall's width.

FIG. 5. Phase propagation and bouncing. The phase (dashed line) and the bubble walls (continuous) are plotted for (a)  $t = 15$ , (b)  $t = 25$ , (c)  $t = 50$  and (d)  $t = 60$ , with  $\gamma = 10$ .

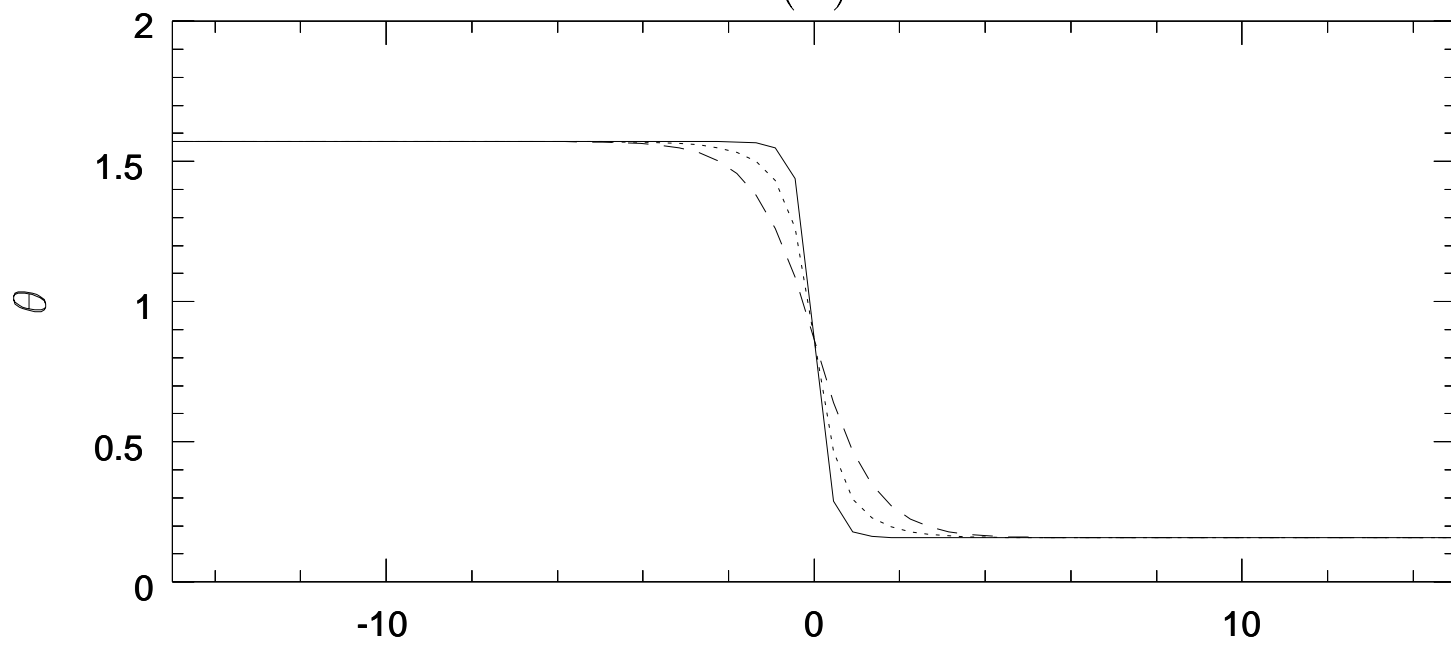
FIG. 6. Collision of two bubbles with phases  $\pi/6$  and  $5\pi/6$ , and subsequent phase interpolation. Continuous lines are contours of equal modulus of the field, dotted lines are phase contours. The phase of the field is also represented by arrows. In this and the following graphs,  $\gamma = 10$ . The axes are arbitrary lattice coordinates.

FIG. 7. Collision of three bubbles with phases  $(\pi/6 - \Delta)$ ,  $5\pi/6 - \Delta$  and  $3\pi/2$ , with  $\Delta = 0.01$ . (a) and (b) are plots of two moments of the collision of two bubbles nucleated at  $t = 0$  with a third one nucleated at  $t = 8$ . (c) and (d) correspond to a similar simulation, where now the third bubble is nucleated at  $t = 18$ , in the same position as the previous one. Only field magnitude contours are represented.

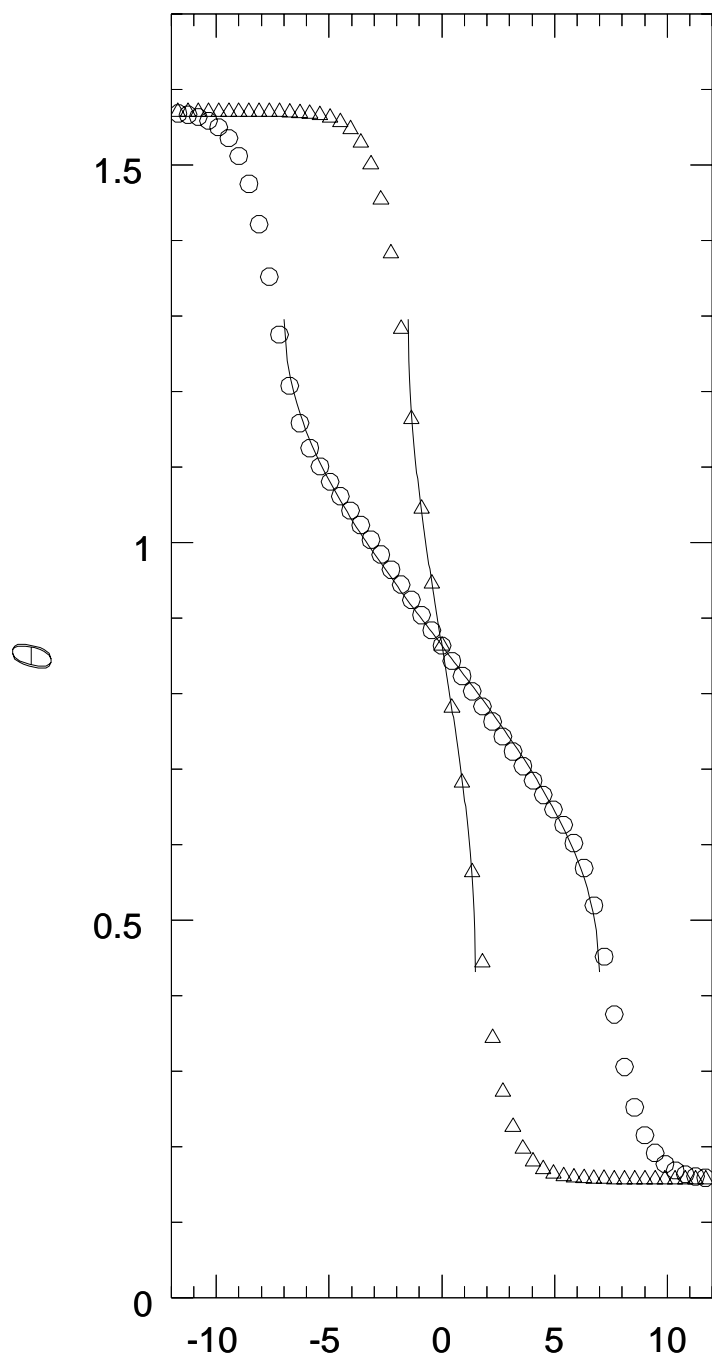
FIG. 8. Later evolution of the system of Figure 6. A third bubble is nucleated at  $t = 48$ . Phase contours are omitted in the last frame for clarity.



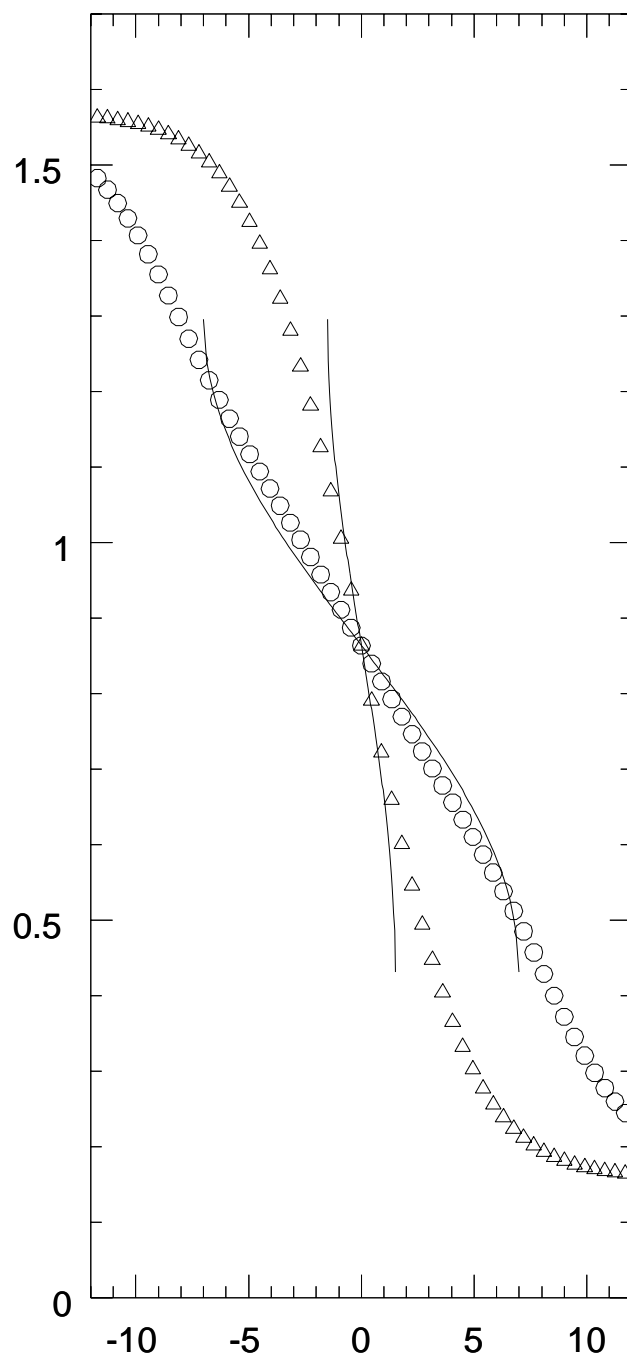
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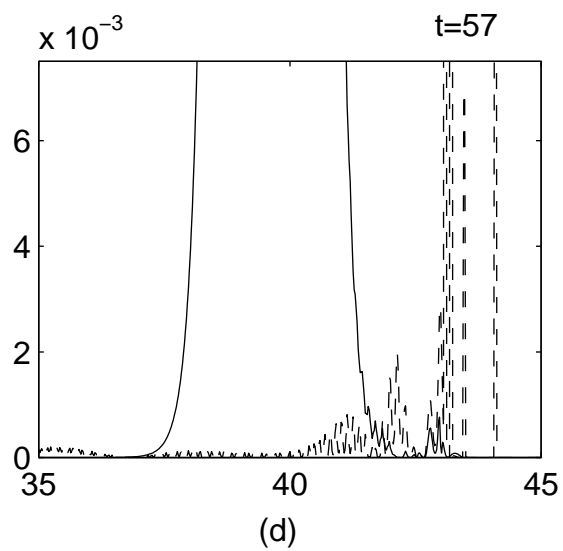
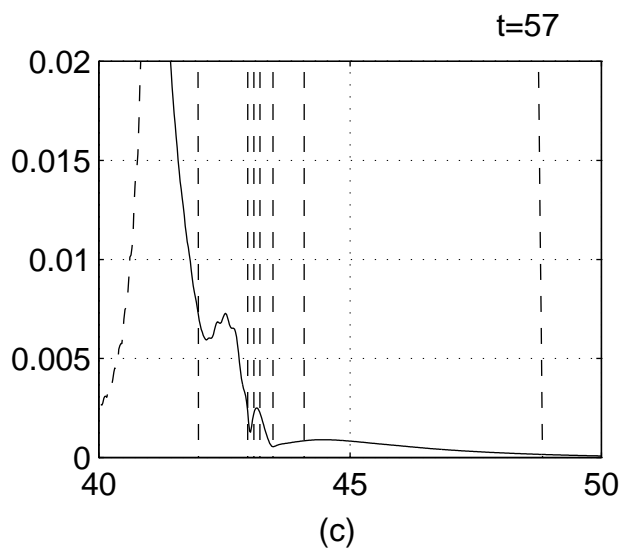
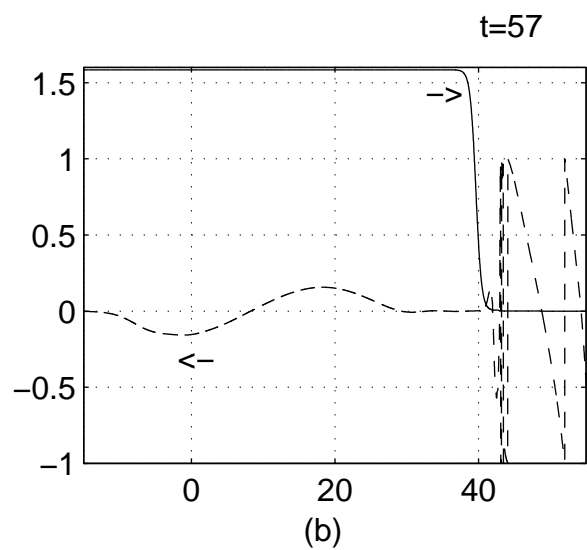
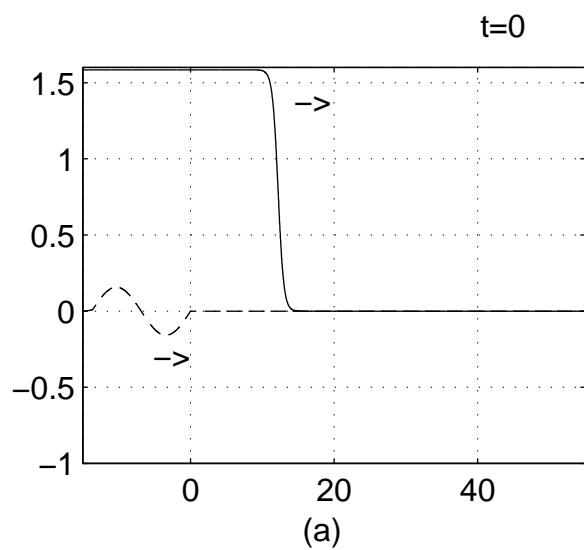
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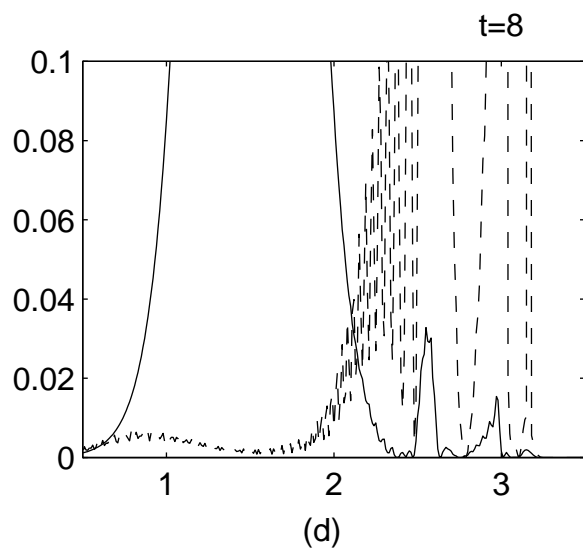
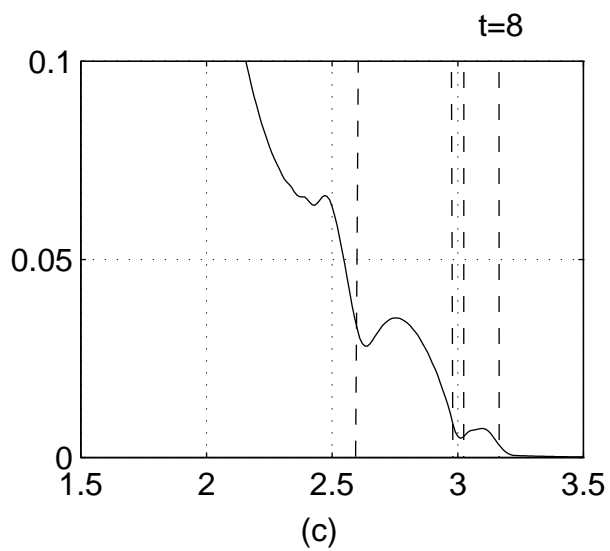
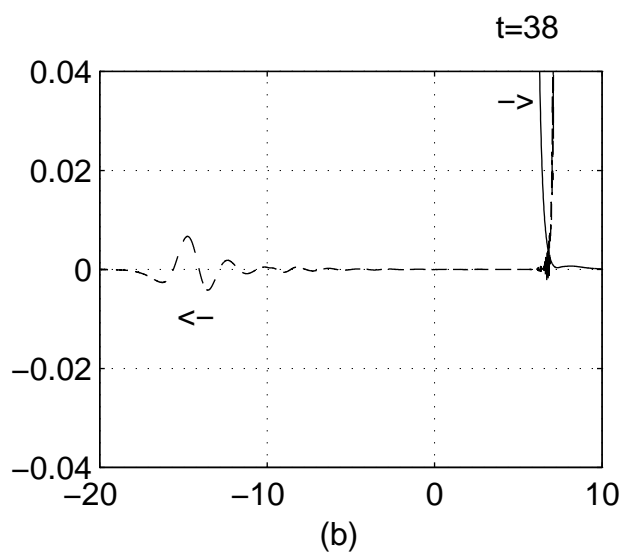
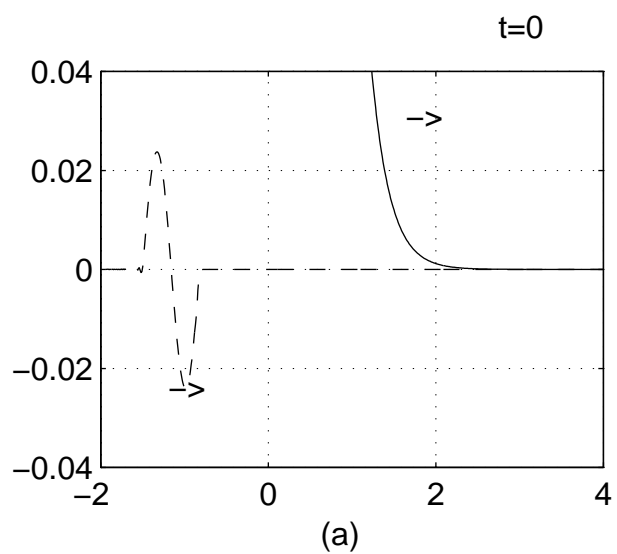


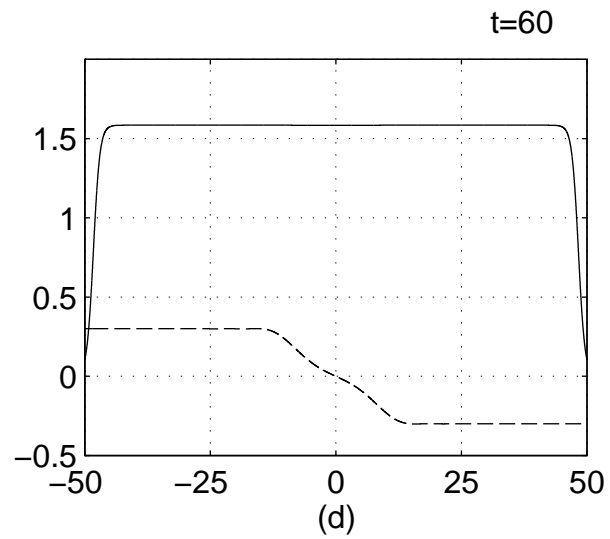
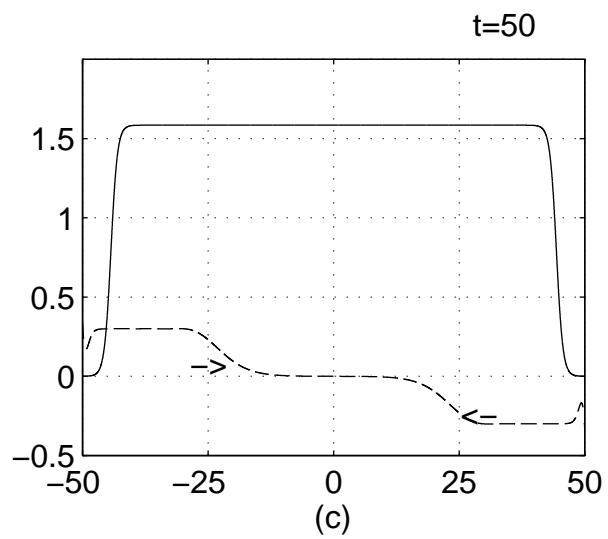
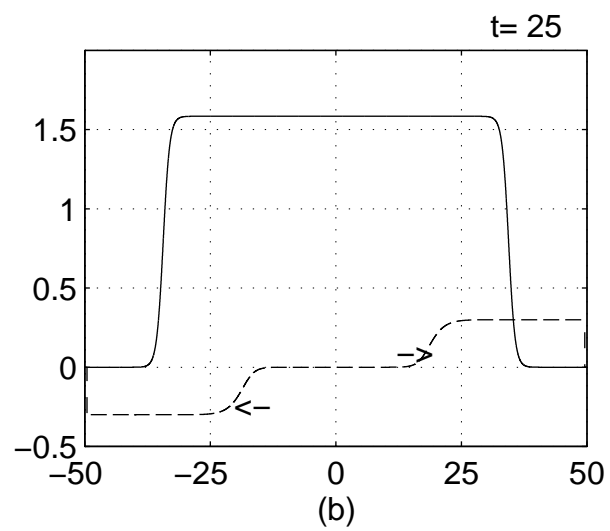
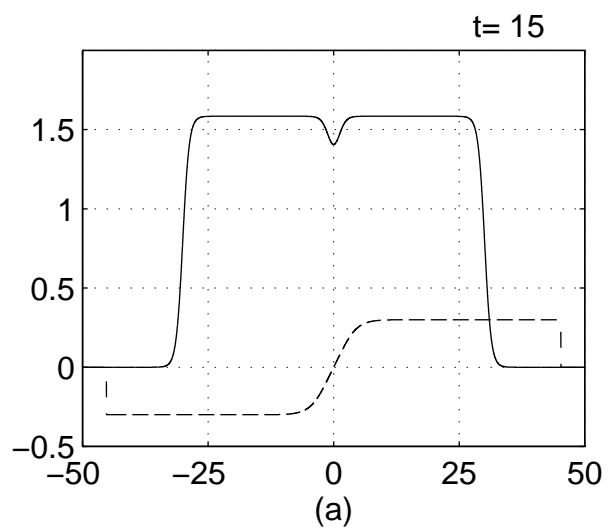
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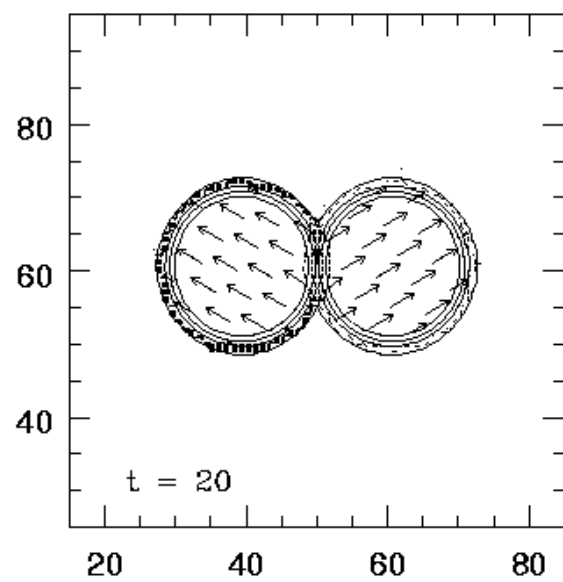
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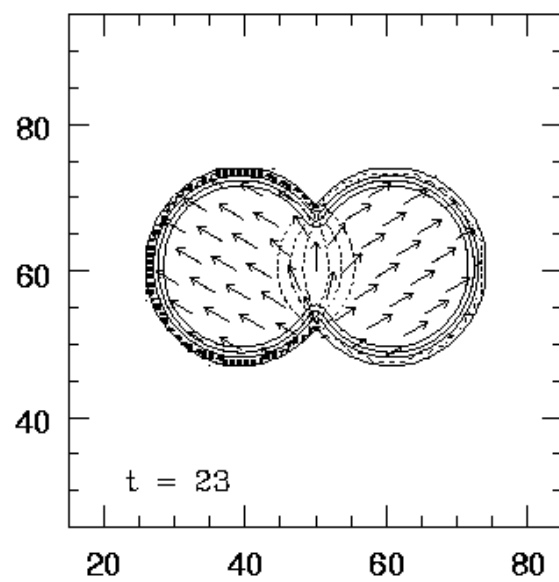




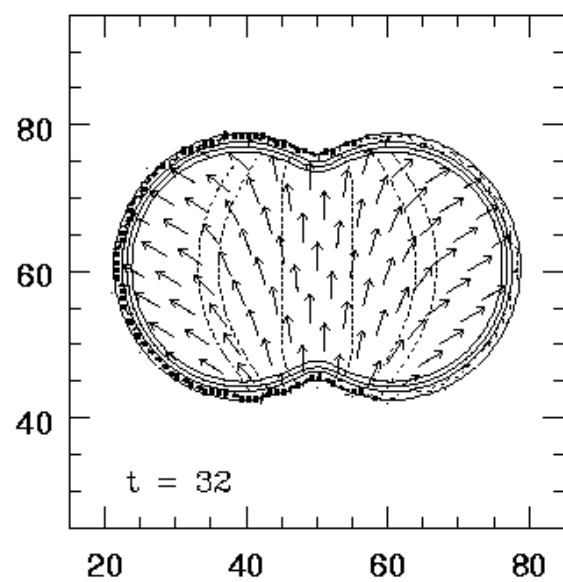




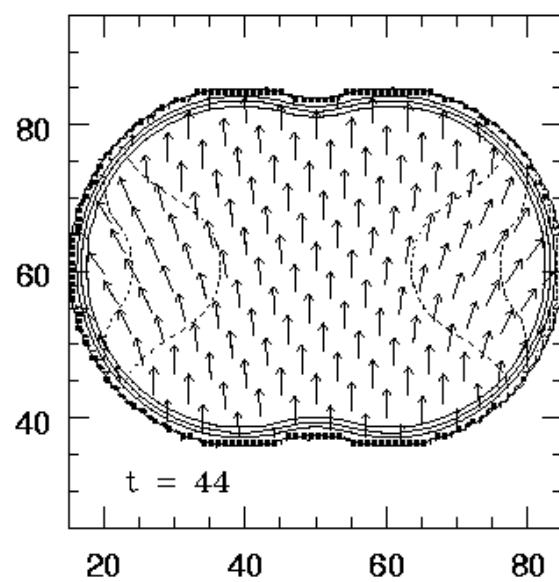
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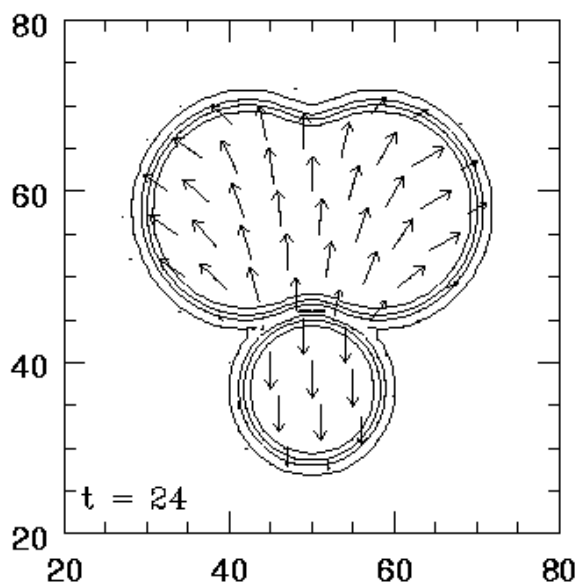
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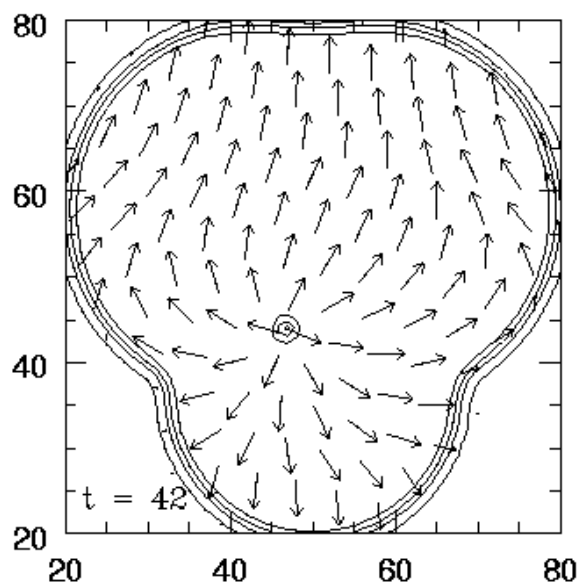
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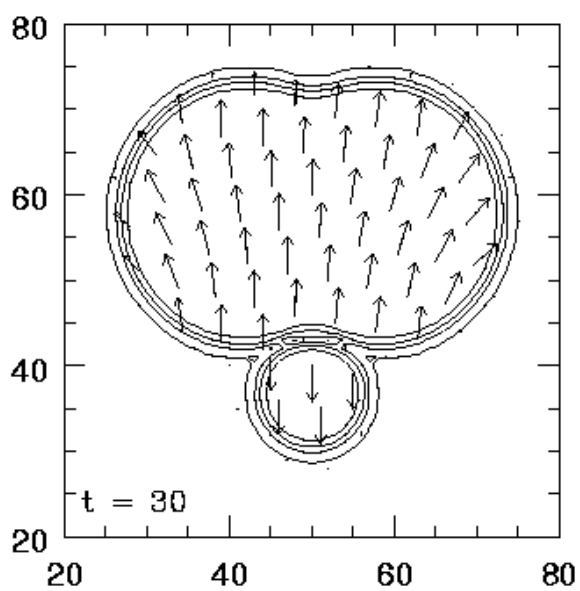
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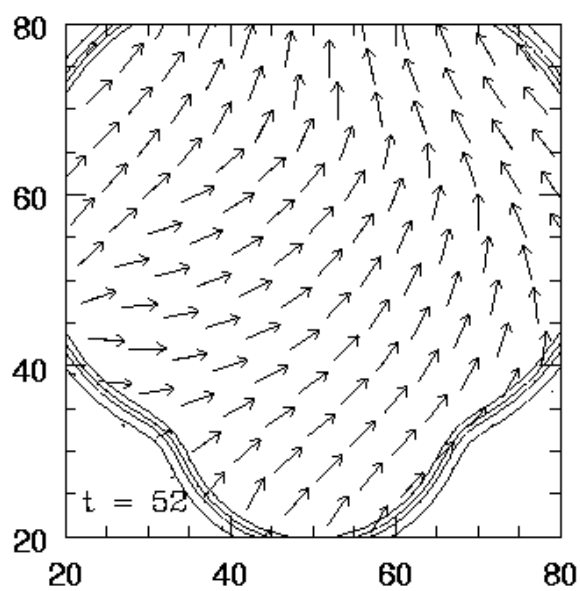
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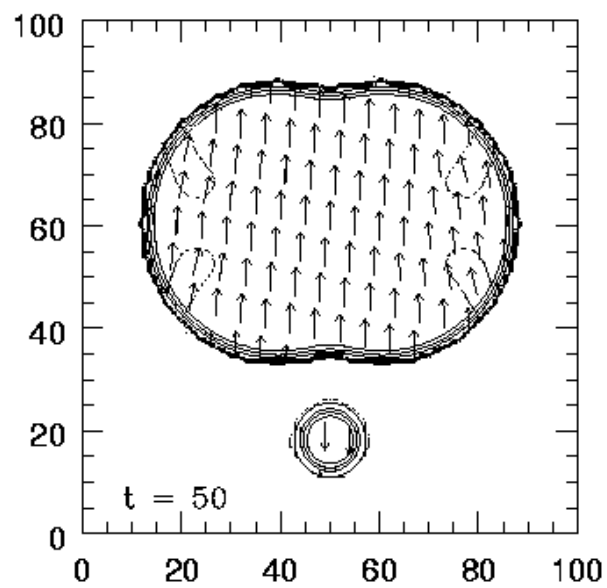
(b)



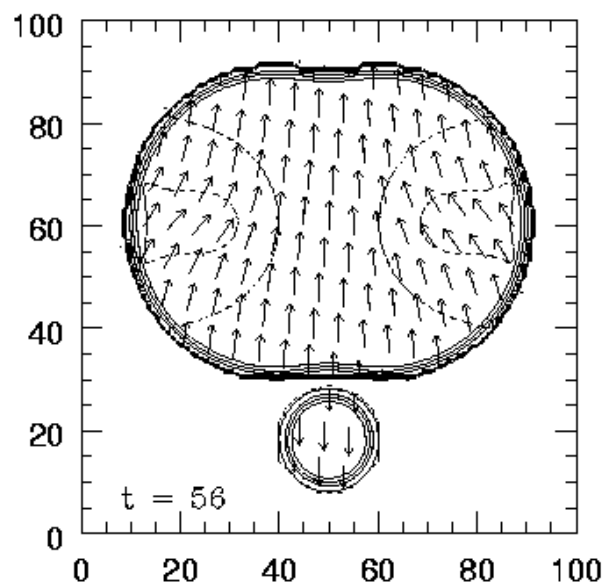
(c)



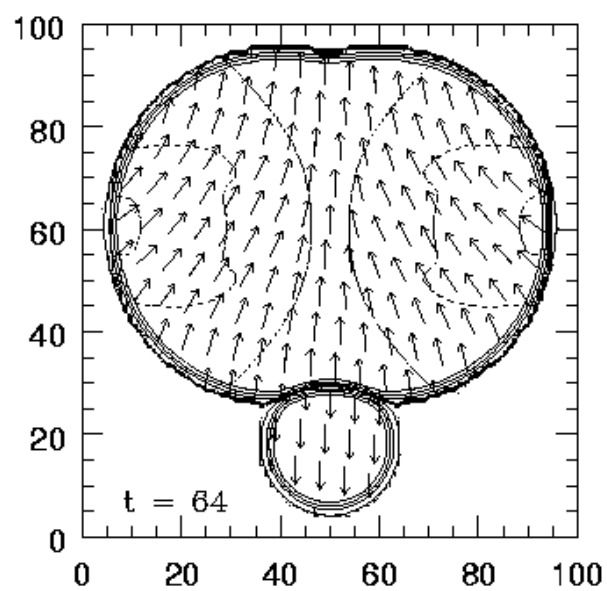
(d)



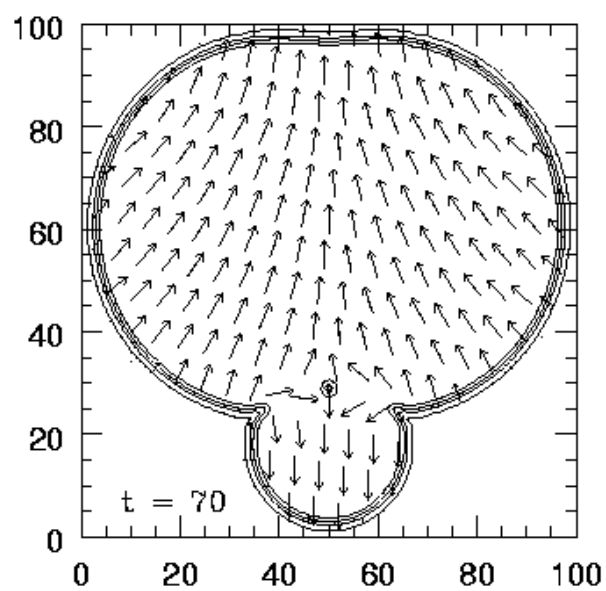
(a)



(b)



(c)



(d)